Syntax of First-Order Logic (FO)

- **Logical symbols**:
  - $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, ( ), $\forall$ ("for all"), $\exists$ ("exists"), ...
- Non-logical symbols: A FO ***signature*** Σ consists of
  - ***constant symbols***: a,b,c, ...
  - ***function symbols***: f, g, ...
  - ***predicate (relation) symbols***: p,q,r, ...

  function and predicate symbols have an associated *arity;*
  - we can write, e.g., p/3, f/2 to denote the ternary predicate p and the function f with two arguments
- First-order ***variables Vars***: x, y, ...
- Formation rules for ***terms*** $\textit{\textbf{Term}}_\Sigma$:
  - constants and variables are *terms*
  - if t_1,...t_k are terms and f is a k-ary function symbol then f(t_1,...,t_k) is a term

Syntax of First-Order Logic (FO)

- Formation rules for *formulas* $\boldsymbol{Fml}_{\Sigma}$:
  - if $t_1,...t_k$ are terms and p/k is a predicate symbol (of arity k) then $p(t_1,...t_k)$ is an *atomic formula* $\boldsymbol{At}_{\Sigma}$ (short: *atom*)
    - all variable occurrences in $p(t_1,...t_k)$ are *free*
  - if F,G are formulas and x is a variable, then the following are formulas:
  - $F \wedge G$, $F \vee G$, $\neg F$, $F \rightarrow G$, $F \leftrightarrow G$, (F),
  - $\forall$**x: F** ("for all x: F(x,...) is true")
  - $\exists$**x: F** ("there exists x such that F(x,...) is true")
  - the occurrences of a variable x within the *scope of a quantifier* are called *bound occurrences.*

Examples

$\forall$x malePerson(x) $\rightarrow$ person(x).

malePerson(bill).

child(marriage(bill,hillary),chelsea).

Variable: x

Constants (0-ary function symbols): bill/0, hillary/0, chelsea/0

Function symbols: marriage/2

Predicate symbols: malePerson/1, person/1, child/2

Semantics of Predicate Logic

- Let ***D*** be a non-empty ***domain*** (a.k.a. *universe of discourse*). A ***structure*** is a pair ***I*** = (***D***,I), with an ***interpretation*** I that maps ...
  - each constant symbols c to an element I(c)$\in$ D
  - each predicate symbol p/k to a k-ary relation I(p) $\subseteq D^k$,
  - each function symbol f/k to a k-ary function I(f): $D^k \rightarrow D$
- Let ***I*** be a structure, β : ***Vars*** $\rightarrow$ ***D*** a *variable assignment*. A valuation $\mathbf{val}_{I,\beta}$ maps $\mathbf{Term}_\Sigma$ to ***D*** and $\mathbf{Fml}_\Sigma$ to {*true*, *false*}
  - $\mathbf{val}_{I,\beta}(x) = \beta(x)$; for x $\in$ ***Vars***
  - $\mathbf{val}_{I,\beta}(f(t_1,\ldots,t_k)) = I(f)(\mathbf{val}_{I,\beta}(t_1),\ldots, \mathbf{val}_{I,\beta}(t_k))$; for $f(t_1,\ldots,t_k) \in \mathbf{Term}_\Sigma$
  - $\mathbf{val}_{I,\beta}(p(t_1,\ldots,t_k)) = I(p)(\mathbf{val}_{I,\beta}(t_1),\ldots, \mathbf{val}_{I,\beta}(t_k))$; for $p(t_1,\ldots,t_k) \in \mathbf{At}_\Sigma$
  - $\mathbf{val}_{I,\beta}(F \wedge G) = \mathbf{val}_{I,\beta}(F)$ and $\mathbf{val}_{I,\beta}(G)$; for F,G $\in \mathbf{Fml}_\Sigma$
  - .... for $\mathbf{Fml}_\Sigma$ over $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, ( ), $\forall$, $\exists$ in the obvious way

Example

Formula F = $\forall x$ malePerson(x) $\rightarrow$ person(x).

Domain D = {b, h, c, d, e}

Let's pick an interpretation $\mathcal{I}$:

$\mathcal{I}$(bill) = b, $\mathcal{I}$(hillary) = h, $\mathcal{I}$(chelsea) = c

$\mathcal{I}$(person) = {b, h, c}

$\mathcal{I}$(malePerson) = {b}

Under this $\mathcal{I}$, the formula F evaluates to *true*.

- If we choose $\mathcal{I}'$ like $\mathcal{I}$ but $\mathcal{I}'$(malePerson) = {b,d}, then F evaluates to *false*
- Thus, $\mathcal{I}$ is a ***model*** of F, while $\mathcal{I}'$ is not:
  - $\mathcal{I}$ |= F $\mathcal{I}'$ |=/= F

FO Semantics (cont'd)

- F ***entails*** G (G is a *logical consequence* of F) if every model of F is also a model of G: F |= G
- F is ***consistent*** or ***satisfiable*** if it has at least one model
- F is ***valid*** or a ***tautology*** if every interpretation of F is a model

Proof Theory:

Let F,G, ... be FO *sentences* (no free variables).

Then the following are equivalent:

1. $F_1, \ldots, F_k \models G$
2. $F_1 \wedge \ldots \wedge F_k \rightarrow G$ is valid
3. $F_1 \wedge \ldots \wedge F_k \wedge \neg G$ is unsatisfiable (inconsistent)

Proof Theory

- A ***calculus*** is formal proof system to establish
  - $F_1, ..., F_k \models G$
- via formal (syntactic) *derivations*
  - $F_1, ..., F_k \vdash ... \vdash G$, where the "$\vdash$" denotes allowed proof steps
- Examples:
  - Hilbert Calculus, Gentzen Calculus, **Tableaux Calculus**, Natural Deduction, Resolution, ...
- First-order logic is "semi-decidable":
  - the set of valid sentences is recursively enumerable, but not recursive (decidable)
- Some inference engines:
  - http://www.semanticweb.org/inference.html

(Semantic) Tableaux Rules

$$\frac{\alpha}{\alpha_1 \atop \alpha_2} \qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

$$\frac{\gamma}{\gamma_1(t)} \qquad \frac{\delta}{\delta_1(c)}$$

$t \in Term^0_\Sigma$

t arbitrary **c new**

- (α) rule for F = **A** $\wedge$ **B**
- (β) rule for F = **A** $\vee$ **B**
- (γ) rule for F = $\forall$**x: A(X,...)**
  - substitute a $\forall$-variable X with an ***arbitrary*** term t
- (δ) rules for F = $\exists$**x: A(X,...)**
  - substitute a $\exists$-variable X with a ***new*** constant **c**

- A ***branch*** *is* ***closed*** if it contains **complementary** formulas
- A ***tableaux*** *is* ***closed*** if every branch is closed

FO Tableaux Calculus

Theorem (Soundness, Completeness of Tableaux calculus):

Let A_1,..., A_k and Th be first-order logic *sentences.*

(Recall: a sentence is a closed formula, i.e., has no free variables)

Then the following are equivalent:

1. $A_1, ..., A_k \models Th$
2. $A_1 \wedge ... \wedge A_k \neg Th$ is unsatisfiable (inconsistent)
3. There is a **closed** tableaux for $\{A_1, ..., A_k , \neg Th\}$

Example

Given:

- (A1) for all x: M(x) ➔ P(x)
- (A2) for all x: P(x) ➔ exists y: c(x,y) and H(y)

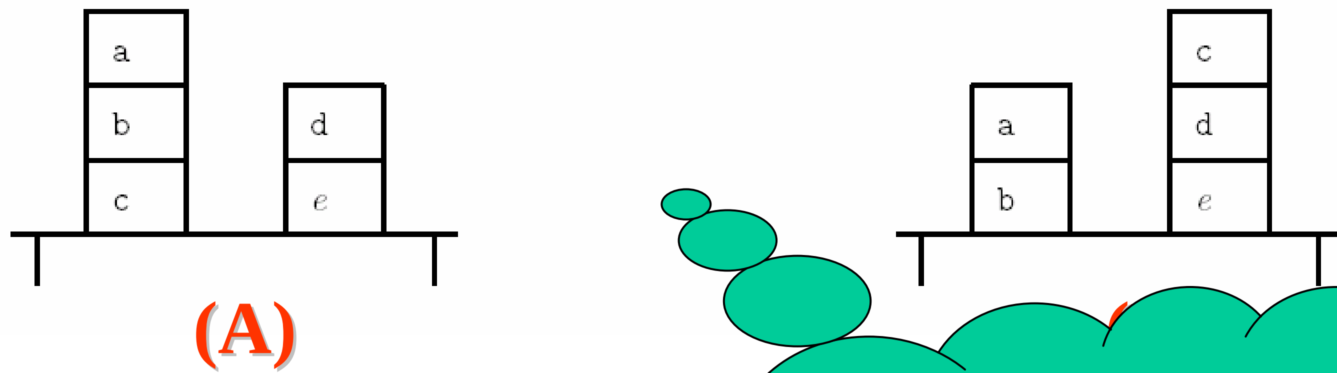
Show:

- (Th) For all x: M(x) ➔ exists y: c(x,y) and H(y)

Proof by contradiction:

- Show that (A1) /\ (A2) /\ **not** (Th) is unsatisfiable

Back to Ontologies: What is a Conceptualization?



Meaning is **NOT** captured by extensional relations i.e. a single state of affairs

- Conceptualization [Genesere
  - universe of discourse (domai
  - relations = {on/2, above/2, clea
- Compare (A) and (B):
  - world_A: {on(a,b), on(b,c), on(d,e),
  - world_B: {on(a,b), on(c,d), on(d,e), table(b), table(e)}
  - two different conceptualizations?
  - or rather two different states of the same conceptualization?

Intensional Structures

- Meaning is ***not*** in a single state of affairs (extensional relations) but can be captured by *intensional relations*
- An n-ary ***intensional relation*** *R* over domain *D* is a function

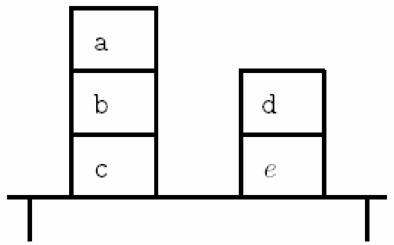
$$R : \quad W \rightarrow \text{Powerset}(D^n)$$

  - *W* set of *possible worlds* {*w1, w2, w3*, ...} (a possible world is one state of affairs, or a *situation*)
  - Powerset(D^n) = set of all subsets of D^n (= D x ... x D)
  - So for each $w \in W$ we have *R(w)* = a subset of D^n, i.e., with each world we associate the interpretation of *R* in that world

Example

- Syntax: *signature (vocabulary)* Σ =
  - – constant symbols: {a,b}
  - – relation symbols: {on/2, table/1}

- Semantics: domain D = {“a_block”, “b_block”}
  Structure *I* = (D,I) with some interpretation I:
  - – I(a) = “a_block”,   I(b) = “b_block”
  - – I(on) = {(I(a),I(b)), (I(b),I(c)), (I(d),I(e))} = {(“a_block”,”b_block”), ...}
  - – I(table) = {c, e}



How can we capture (some of!) the meaning of “on-ness”?

- Many things can be said about “on-ness” (physics of gravity, pressure and deformation, etc.)
- What is common among all possible states of on/2 over a certain domain D?
- That is, if we look at all possible worlds W, and the values that I(on)(w) can take, what is common among all those states?
- What is *always* true (in all possible worlds) about on/2 is (part of) the meaning of on/2.
  - ($\forall$x: $\neg$ on(x,x)) ; in all possible worlds: x is not on x
  - ($\forall$x,y: $\neg$ (on(x,y) $\wedge$ on(y,x))) ; in all possible worlds: no x is on y while y is on x
  - – Good enough? what about on(a,b), on(b,c), on(c,a) ?
  - – Even worse: What if someone sees “on” and understands/interprets it as “below”?

  ➔ we only capture *some* aspects using the above ontological theory