CSE-291 Ontologies in Data and Process Integration
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Last Week

Temporal Databases and Statelog

- Add an extra argument to predicates indicating the valid time (point)
- time isomorphic to \((\mathbb{N}_0, \leq)\) or \((\mathbb{Z}, \leq)\)
- point-based (as opposed to interval based) temporal model

Temporal database \( \mathcal{D} = (\mathcal{D}[0], \mathcal{D}[1], \mathcal{D}[2], \ldots) \)

with this define, e.g.,
\[
\mathcal{D} \models \square F \text{ if for all } n \in \mathbb{N}_0 : \mathcal{D}[n] \models F
\]
\[
\mathcal{D} \models \Diamond F \text{ if for some } n \in \mathbb{N}_0 : \mathcal{D}[n] \models F
\]

Applications: e.g., active databases, temporal databases, temporal knowledge representation and reasoning
Temporal logics are special modal logics.

In classical logic, it is only important whether a formula is true.

In modal logic, it is also important in which way, mode, or state a formula is true.

A formula (proposition) can be

- necessarily or possibly true
- true at some time point and false at another
- believed or known

...
The basic modal operators are usually given to be possibility, actuality, and necessity.

A sentence is said to be **actual** if it is true; it is said to be **possible** if it might be true (whether it is actually true or actually false).

A **necessary** statement is one which could not possibly be false.

Also, the term contingent will come up with regard to modality. A **contingent** truth is one which is possibly true, and possibly false. (This is not the same, of course, as saying that it is a statement which might be both true and false; there are no statements of that sort.)
Modal logic is most often used for talk of so-called alethic modalities: "it is necessarily the case that..." or "it is possibly the case that...."

These (also called metaphysical modalities or subjunctive modalities) need to be distinguished from similar-sounding claims using epistemic modalities.

E.g., when a philosopher claims that Bigfoot possibly exists, he probably does not mean that "it’s possible that Bigfoot exists—for all I know." Rather, he makes the metaphysical claim that "it’s possible for Bigfoot to exist"—which is a substantive claim concerning ways the world could have been.
Digression: What is Metaphysics?

The term metaphysics is derived from the Greek phrase *Ta Meta ta Physkia* which simply means "the books after the books on nature."

When a librarian was cataloging Aristotle’s works, he did not have a title for the material which he wanted to shelve after the material called "nature" (Physkia) – so he simply called it "after nature."

Originally, this wasn’t even a subject at all – it was just a collection of notes on different topics, but specifically topics which were removed from normal sense perception and empirical observation.
... In Western philosophy, metaphysics has become the study of the fundamental nature of all reality – what is it, why is it, and how are we to understand it.

Some only regard metaphysics as the study of "higher" reality or the "invisible" nature behind everything, but that isn’t actually true. It is, instead, the study of all of reality, visible and invisible.

Source: Austin Cline – Metaphysics: What is it?
... Conversely, we might say "Goldbach’s Conjecture is possibly true, but possibly it is false."

Here we mean that it is *epistemically possible* that it is true or false.

At the same time if the sentence is true it is *logically necessary* that it is true, and if it is false it is *logically impossible*. 
Several modes of speech are closely related, e.g., about time. It seems reasonable to say that possibly it will rain tomorrow, and possibly it won’t; on the other hand, if it rained yesterday (if it really did rain), then it cannot be quite correct to say "It may not have rained yesterday." It seems the past is "fixed," or necessary, in a way the future isn’t. Many philosophers and logicians think this reasoning isn’t very good; but the fact remains that we often talk this way and it is good to have a logic to capture its structure. Likewise talk of morality, or of obligation generally, seems to have a modal structure. The difference between "You must do this" and "You may do this" looks a lot like the difference between "This is necessary and this is possible."

Such logics are called \textit{deontic}, from the Greek for "duty". 
Significantly, modal logics can be developed to accommodate most of these idioms; it is the fact of their common logical structure (the use of "intensional" or non-truth-functional sentential operators) that make them all varieties of the same thing.

Epistemic logic is (arguably) best captured in the system $\mathbf{S}_4$; deontic logic in the system $\mathbf{D}$, temporal logic in $\mathbf{t}$ and alethic logic $\mathbf{S}_5$. 
... On the other hand, suppose that someone asks you if 54 squared is 2926 and you stammer, "I don’t know, I suppose it’s possible." Here you are using an epistemic possibility—you are saying that "For all I know, it’s possible that 54 squared is 2926." But you are almost surely not making the very hasty claim that it’s *metaphysically possible* for 54 squared to be 2926—which is fortunate, since it turns out that 54 squared is 2916, and it’s *metaphysically impossible* for it to have been otherwise.
Introduction to (First-Order) Modal Logic

Modal operators are applied to formulas:
- atomic formulas are formulas

- if $A$, $B$ are formulas ($x$ a variable) then so are:
  - $A \land B$
  - $A \lor B$
  - $\neg A$
  - $A \rightarrow B$
  - $A \leftrightarrow B$
  - $\forall xA$
  - $\exists xA$
  - $\Box A$ (first-order quantifiers)
  - $\Diamond A$ ("necessarily $A$")
  - $\Diamond A$ ("possibly $A$")
Different modal logics are obtained
(a) by “reading”/interpreting the modal operators differently
(b) by axiomatizing different accessibility relations $R$ between possible worlds accordingly

(a) for example:
- $\Box A$: “necessarily $A$” or “always in the (future/past) $A$”, ...
- $\Diamond F$: “$F$ is possible”
  equivalent to $\neg \Box \neg F$: “not necessarily not $F$”

(b) The following axiom captures that $R$ is transitive (i.e., if $R(x, y)$ and $R(y, z)$ then $R(x, z)$):
- $\Box F \rightarrow \Box \Box F$
# Introduction to Modal Logic

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\square F$</td>
<td>necessarily $F$</td>
</tr>
<tr>
<td>$\Diamond F$</td>
<td>possibly $F$</td>
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<tbody>
<tr>
<td>$\square F$</td>
<td>always (in the future/past) $F$</td>
</tr>
<tr>
<td>$\Diamond F$</td>
<td>sometimes (in the future/past) $F$</td>
</tr>
<tr>
<td>$\Diamond F$</td>
<td>$F$ is consistent with $a$’s beliefs</td>
</tr>
<tr>
<td>$\square F$</td>
<td>$F$ is true after all possible executions of $p$</td>
</tr>
<tr>
<td>$\Diamond F$</td>
<td>$F$ is true after some possible execution of $p$</td>
</tr>
</tbody>
</table>

To indicate which “reading” is meant, write e.g.

- $\square_{a_{if}} F$
- $\square_{a_{ip}} F$
- $\square_{a}^{(k|b)} F$
- $[p] F$
- ...
A Puzzle

Each of three wise men is put a hat on his head; either a white one or a black one. There is at least one black hat. Everyone sees the other ones’ hats but not their own.

After some time one says: “I don’t know what hat I have”.

Then another one says: “I don’t know either”.

Then the third one says: “I know that I have a black one!”. 
Formalizing the Puzzle

Notation

- $B_i$: agent (wise man) $i$ has a black hat
- $W_i$: agent $i$ has a white hat
- $\Box_i F$: agent $i$ knows $F$
- $\Diamond_i F$: agent $i$ doesn’t know that $F$ is false

for $i = 1, 2, 3$
Formalizing the Puzzle

Facts

- $\neg \Box_1 B_1$  
- $\neg \Box_2 B_2$  
- $W_i \land W_j \rightarrow B_k$ for $i, j, k \in \{1, 2, 3\}$ and pairwise different  
- $W_i \rightarrow \Box_j W_i$ for $i, j, k \in \{1, 2, 3\}, i \neq j$  
- $\neg W_i \rightarrow \Box_j \neg W_i$ for $i, j, k \in \{1, 2, 3\}, i \neq j$  
- $\neg W_i \rightarrow B_i$ for $i \in \{1, 2, 3\}$  
- $\neg B_i \rightarrow W_i$ for $i \in \{1, 2, 3\}$  
- $B_i \lor W_i$ for $i \in \{1, 2, 3\}$
Acceptable Inference Rules

- All tautologies of propositional logic, e.g.

\[ \square_i A \lor \neg \square_i A \]

- Good old modus ponens:

\[ \begin{array}{c}
A \\
A \rightarrow B
\end{array} \quad \Rightarrow \\
\begin{array}{c}
B
\end{array} \]

- The wise men know the rules of the game, so:

\[ \begin{array}{c}
A \\
\square A
\end{array} \]

(here: \( \square \) is any of the \( \square_i \)) (G)
... Acceptable Inference Rules

- if I know $A$ and I know that $A$ implies $B$, then I know $B$

if $\Box A$ and $\Box (A \rightarrow B)$ then $\Box B$  

(K)

- ... or equivalently (why?)

$(\Box A \land \Box (A \rightarrow B)) \rightarrow \Box B$  

(K)

$\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$  

(K)
Example Inference

Show that

\[
\begin{align*}
A \rightarrow B \\
\Box A \rightarrow \Box B
\end{align*}
\]
Modal Logic Inference System

A modal logic system $\mathcal{S}$ consists of axioms and inference rules.

Such a system induces a provability relation $\vdash_\mathcal{S}$

$$\Sigma \vdash_\mathcal{S} A$$

holds for a set of formulas $\Sigma$ and formula $A$ (here: both propositional modal logic), if there exists a proof of $A$ using only the formulas $\Sigma$, the axioms, and the allowed inference rules.
The Modal Logic System K

Axioms
- all propositional axioms
- \((A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)\) \hspace{1cm} (K)

Inference Rules
- Modus Ponens:

\[
\frac{A \hspace{1cm} A \rightarrow B}{B}
\]

- Generalization:

\[
\frac{A}{\Box A}
\] 

(G)
Some Modal Logic Systems

- **T:** $K$ plus $\Box A \rightarrow A$
- **D:** $K$ plus $\Box A \rightarrow \Diamond A$
- **B:** $T$ plus $\neg A \rightarrow \Box \neg \Box A$
- **S4:** $T$ plus $\Box A \rightarrow \Box \Box A$
- **S5:** $T$ plus $\neg \Box A \neg \Box A$
- **...**
Kripke Structures and Puzzle Resolved

[[folien15: 14–18; 25–31]]
Lemma   A Kripke Frame \((G, R)\) is reflexive iff in all Kripke structures \(K = (G, R, v)\) the formula \(\square p \rightarrow p\) is valid.

\((G, R)\) is called reflexive if \(R\) is reflexive, i.e.,
\((G, R) \vDash \forall x : R(x, x)\)

Proof   Assume \((G, R)\) is reflexive and consider an arbitrary valuation \(v : G \times P \rightarrow \{0, 1\}\). For every \(g \in G\) we have to show that \((K, g) \vDash \square p \rightarrow p\). Clear if the premise is false, i.e.,
\((K, g) \nvDash \square p\). Else if \((K, g) \vDash \square p\) then – by definition of \(\square\) – for all \(h \in G\) such that \(R(g, h)\) we have \((K, h) \vDash p\). Since \(R\) is reflexive, specifically \((K, g) \vDash p\).

Now assume \((G, R)\) is not reflexive. Pick \(g_0 \in G\) with \(\neg R(g_0, g_0)\). Define \(v_0(g, p) := 1\) if \(R(g_0, g)\) and 0 otherwise. For \(K_0 = (G, R, v_0)\) with have \((K_0, g_0) \vDash \square p\) but also \((K_0, g_0) \vDash \neg p\). Hence \(\square p \rightarrow p\) is not valid in \(K_0\).