Review: Set Operation and Subset

• Intersection: \( A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \)

• Union: \( A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \)

• Cartesian Product (also called product): \( A \times B = \{ (x, y) \mid x \in A, y \in B \} \)

• The set of all subsets of \( A \): \( P(A) \). \(|P(A)|=2^{\mid A \mid}\)

• The set of all subsets of \( A \) of size \( k \): \( P_k(A) \). \(|P_k(A)|=C(\mid A \mid, k)\)
Learning Outcomes

• By the end of this lesson, you should be able to
  – Understand function concept and use its different notations correctly.
  – Understand the different types of functions, tell which function type(s) a real function belongs to.
  – Use function diagram to help understanding and the judgment.
Why do we need to learn them?

• Function plays a fundamental role in nearly all of Mathematics.

• In Computer Science
  – In almost all programming languages, we use function for a certain task, e.g., weatherForcast(location).
  – There is a special programming paradigm called functional programming.
More Notations of Set

• The set of the first $n$ positive integers, \{1, 2, \ldots, n\} : n.

• Linear order the elements of a set $A$ using a list: $(a_1, a_2, \ldots, a_{|A|})$ or $a_1, a_2, \ldots, a_{|A|}$.

• By default, the ordering on a set of numbers is the numerical ordering. For example, the ordering on $n$ is $1, 2, 3, \ldots, n$. 
Definition 1: Function

• If $A$ and $B$ are sets, a function from $A$ to $B$ is a rule that tells how to find a unique $b \in B$ for each $a \in A$.

• $f : A \rightarrow B$ means $f$ is a function from $A$ to $B$.

• We call the set $A$ the domain of $f$ and the set $B$ the range/codomain of $f$.

• To specify a function completely, you must give its domain, range and rule.
More Notations of Function

• Definition 2 (*One-line notation*): when $A$ is ordered by $a_1, a_2, \ldots, a_{|A|}$, a function can be written in one-line notation: $(f(a_1), f(a_2), \ldots, f(a_{|A|}))$

• The set of all functions from $A$ to $B$: $B^A$
  
  – $f : A \rightarrow B$ equals $f \in B^A$.
  
  – from one-line notation, each function is an element of Cartesian product of $|A|$ number of set $B$: $B \times B \times \ldots \times B$.
  
  – total function number from $A$ to $B$ is the size of the product: $|B|^{|A|}$. 
Function Examples

- \( P = \{a, b, c\} \), \( g : P \rightarrow 4 \), \( g(a) = 3 \), \( g(b) = 1 \) and \( g(c) = 4 \)
  - function name: \( g \)
  - domain: \( \{a, b, c\} \)
  - range: \( \{1, 2, 3, 4\} \)

- Equivalent expression: \( g : 4^{\{a, b, c\}} \), ordering: \( a, b, c \), \( g = (3, 1, 4) \)

- How many functions we can have with the same domain and range? \( |4^{|\{a,b,c\}|}| = 4^3 \)
Diagram of Function

- $P = \{a,b,c,d\}$, $g: P \rightarrow 4$, $g(a)=3$, $g(b)=1$, $g(c)=4$ and $g(d)=1$. 
Definition 3 : Types of Functions

- Let \( f : A \rightarrow B \) be a function.
  - \( f \) is a **surjection**: for every \( b \in B \), there is an \( a \in A \) such that \( f(a) = b \). It also means \( f \) reaches each value in its range at least once.
  - \( f \) is an **injection** (one-to-one function): \( f(x) = f(y) \) implies \( x = y \). It also means \( f \) reaches each value in its range at most once.
  - \( f \) is a **bijection**: \( f \) is both an injection and a surjection.
Example 3: Types of Functions

- let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ be the domain, and range of the function $f = (a, b, a)$
  - surjection; not injection
- function $g$ with domain $B$ and range $A$ given by $g(a) = 3$ and $g(b) = 1$
  - injection; not surjection
- the function $h$ with domain $B$ and range $C = \{1, 3\}$ given by $h(a) = 3$ and $h(b) = 1$
  - bijection
Definition 3: Types of Functions (2)

- **Permutation**
  - For a function to itself, such as \( f : A \rightarrow A \), if this function is a bijection, it is called a *permutation*.
  - All bijections from \( A \) to \( A \) are called *permutations* of \( A \).
  - For set \( A \), its permutation number is \(|A|!\).

- **Inverse function**: if \( f : A \rightarrow B \) is a bijection, we can have a new function called the *inverse* of \( f \), written as \( f^{-1} \), which reverses what \( f \) does.
  - \( f^{-1} : B \rightarrow A \) and \( f^{-1}(b) \) is that unique \( a \in A \) such that \( f(a) = b \).
  - \( f(f^{-1}(b)) = b \) and \( f^{-1}(f(a)) = a \).
Example of Permutation and Inverse Function

- let $A = \{1, 2, 3\}$, list permutations of $A$.
  - $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2),(3,2,1)$

- The function $h$ with domain $B = \{a, b\}$ and range $C = \{1, 3\}$ given by $h(a) = 3$ and $h(b) = 1$. Specify its inverse function if there is one.
  - Because $h$ is a bijection, it has inverse function.
  - $h^{-1}$ has domain $C = \{1, 3\}$ and range $B = \{a, b\}$, and $h^{-1}(3) = a$ and $h^{-1}(1) = b$.  

Example 4: Functions as Relations

- Let $A$ and $B$ be sets. A *relation* from $A$ to $B$ is a subset of $A \times B$.
  
  If $A = 3$ and $B = 4$, then $R = \{(1,4),(1,2),(3,3),(2,3)\}$ is a relation from $A$ to $B$.

- *functional relation*: If the relation $R$ satisfies the condition that, for all $x \in A$ there is a unique $y \in B$ such that $(x, y) \in R$, then the relation $R$ is called a functional relation.

- A functional relation defines a function.
Example 5 : Two-line Notation

• The first line lists the domain elements, and the second line lists the values of these domain elements.

• For \( X=\{x_1, x_2, \ldots, x_n\} \), \( Y=\{y_1, y_2, \ldots, y_n\} \), function \( f: f(x_1)=y_1, f(x_2)=y_2, \ldots, f(x_n)=y_n \)

\[
f = \begin{pmatrix}
x_1 & x_2 & \ldots & x_n \\
y_1 & y_2 & \ldots & y_n
\end{pmatrix}
\]

• Some range elements might not be shown in this notation because the cardinality/size of \( Y \) can be greater than that of \( X \).
Example 5: Two-line Notation (2)

• The function $f$ with domain and range \{a, b, c, d\} given in 2-line form by
  \[
  f = \begin{pmatrix}
  a & b & c & d \\
  b & c & a & d
  \end{pmatrix}
  \]

• What are the types of this function?
  – surjection, injection, bijection

• Is it a permutation?
  – yes, because the domain and the range are the same set.

• Does it have an inverse function? Write the function if it has.
  – It has an inverse function.
  \[
  f^{-1} = \begin{pmatrix}
  b & c & a & d \\
  a & b & c & d
  \end{pmatrix}
  \]
Homework and Pre-Reading Assignment

• Homework:
  – Exercise 1.1, 1.2, 1.3, 1.4, in page Fn-6 to Fn-7

• For next class, please read Section 2 (Fn-7 to Fn-10).
  – Try to understand function composition through examples.
  – Try to understand operations of permutation through examples.